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Monotonic Solutions of Urysohn Integral Equation on Unbounded Interval

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Abstract—In this paper, we investigate a nonlinear Urysohn integral equation on unbounded interval. We show that under some assumptions that the equation has monotonic solutions belonging to the space of functions being Lebesgue integrable on unbounded interval. The main tool used in our study is the technique associated with measures of weak noncompactness and measures of noncompactness in strong sense.

Keywords—Urysohn integral equation, Measure of noncompactness, Compactness in measure, Weak sequential continuity.

1. INTRODUCTION

One of the most frequently investigated equation in nonlinear analysis is the famous Urysohn integral equation, having the form

$$x(t) = f(t) + \int_I u(t, s, x(s))\, ds,$$

where I is an interval in $\mathbb{R}$ (bounded or not), $t \in I$ and the functions $f : I \to \mathbb{R}$, $u : I \times I \times \mathbb{R} \to \mathbb{R}$ are given, while $x : I \to \mathbb{R}$ is an unknown function. The case when I is bounded interval is rather classical. Indeed, in this case the theory of the above equation is well developed (cf. [1–4] and references therein).

On the other hand, the case when I is an unbounded interval is more difficult and complicated. Therefore, in this paper, we focus on this situation assuming, for simplicity, that $I = \mathbb{R}_+ = [0, \infty)$.

Thus, the aim of this paper is to discuss the solvability of the nonlinear integral equation of Urysohn type having the form

$$x(t) = f(t) + \int_0^\infty u(t, s, x(s))\, ds, \qquad t \geq 0. \tag{1.1}$$

Using the technique associated with measures of noncompactness (both in strong and weak sense), we prove the existence of solutions of equation (1.1) in the space $L^1(\mathbb{R}_+)$ of Lebesgue integrable functions on the halfaxis $\mathbb{R}_+$.

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The approach applied in this paper depends on extending of the methods and tools used in the study of some nonlinear integral equations which are presented in papers [3–6], among others. Let us notice that in those papers, the solvability of the mentioned integral equations was considered in the space of Lebesgue integrable functions on a bounded interval I.

2. NOTATION AND SOME AUXILIARY RESULTS

At the beginning of this section, we present a few facts concerning measures of noncompactness. The notion of the measure of noncompactness (in strong sense) presented here comes from [7].

Assume that E is a Banach space with the norm $\|\cdot\|$ and zero element Θ. For a set $X \subset E$ denote by $\bar{X}$ the closure of X and by $\bar{X}^w$ the weak closure of X. The symbol $\operatorname{Conv} X$ stands for the convex closure of a set X. The symbol $B(x, r)$ denotes the closed ball centered at x and with radius r. We write B_r instead of $B(\Theta, r)$ and B_E instead of B_1.

Further, denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$, $\mathfrak{N}_E^w$ its subfamilies consisting of all relatively compact and relatively weakly compact sets, respectively.

DEFINITION 2.1. *(See [7].) A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness provided it satisfies the following conditions.*

- 1^o *The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.*
- 2^o $X \subset Y \Longrightarrow \mu(X) = \mu(Y)$.
- 3^o $\mu(\operatorname{Conv} X) = \mu(X)$.
- 4^o $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$, $\lambda \in [0, 1]$.
- 5^o *If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$, such that $X_{n+1} \subset X_n$, for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.*

The family $\ker \mu$ described in 1^o is called the *kernel of the measure μ*.

DEFINITION 2.2. *(See [8].) A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is referred to as a measure of weak noncompactness if it satisfies conditions $2^o, 3^o, 4^o$ of Definition 2.1 and the following two conditions (being counterparts of 1^o and 5^o):*

- $1^{o\prime}$ *the family $\ker \mu$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$;*
- $5^{o\prime}$ *if (X_n) is a sequence of weakly closed sets from $\mathfrak{M}_E$, such that $X_{n+1} \subset X_n$, for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the set X_∞ is nonempty.*

One of the most important and useful measure of noncompactness is the *Hausdorff measure* χ defined as follows [7]:

$$\chi(X) = \inf\{\varepsilon > 0 : \text{there exists a finite subset } Y \text{ of } E \text{ such that } X \subset Y + B_\varepsilon\}.$$

Let us also recall that the first important and convenient measure of weak noncompactness β has been defined by De Blasi [9]

$$\beta(X) = \inf\{\varepsilon > 0 : \text{there exists a weakly compact subset } Y \text{ of } E \text{ such that } X \subset Y + B_\varepsilon\}.$$

The measures χ and β have a lot of interesting and useful properties [7–9].

Now, we recall the fixed-point theorem of Darbo type [7,10] which is very useful tool in applications and will be used further on.

THEOREM 2.1. *Let Ω be a nonempty, bounded, closed, and convex subset of E and let $F : \Omega \to \Omega$ be a continuous mapping which is a contraction with respect to a measure of noncompactness μ, i.e., there exists a constant $k \in [0, 1)$ such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset X of Ω. Then F has at least one fixed point in the set Ω.*

REMARK 2.1. Theorem 2.1 remains valid if μ is a measure of weak noncompactness and if we assume that F is a weakly continuous (or even sequentially weakly continuous) transformation (cf. [11,12]).

Now, let $I \subset \mathbb{R}$ be a fixed interval, bounded or not. Let m be the Lebesgue measure on I. Denote by $S = S(I)$ the set of all Lebesgue measurable real functions defined on I.

The set S furnished with the metric

$$\rho(x, y) = \inf[a + m(\{s : |x(s) - y(s)| \geq a\}) : a \geq 0]$$

becomes a complete metric space [13]. Moreover, it is well known that convergence in measure on I coincides with convergence generated by the metric ρ.

Further, let $L^1(I)$ denote the space of Lebesgue integrable on I functions, normed in the standard way

$$\|x\| = \|x\|_{L^1(I)} = \int_I |x(t)|\, dt.$$

Recall that the complete description of compactness in measure (i.e., compactness in the space $S(I)$) was given by Fréchet [13]. For our purposes, it is sufficient to recall the following particular case of the Fréchet result [5,6].

THEOREM 2.2. *Let X be a bounded subset of $L^1(I)$ consisting of functions which are a.e. nondecreasing (or nonincreasing) on the interval I. Then I is compact in measure.*

In the sequel, we will work in the space $L^1(\mathbb{R}_+)$ which will be denoted shortly by L^1. We recall the formula for a measure of weak noncompactness [14]. Namely, fix a bounded subset X of L^1 and define

$$c(X) = \lim_{\varepsilon \to 0} \left\{ \sup_{x \in X} \left\{ \sup \left[\int_D |x(t)|\, dt : D \subset \mathbb{R}_+,\ m(D) \leq \varepsilon \right] \right\} \right\},$$

$$d(X) = \lim_{T \to \infty} \left\{ \sup \left[\int_T^\infty |x(t)|\, dt : x \in X \right] \right\}.$$

Finally, let us put

$$\gamma(X) = c(X) + d(X).$$

Then, we have the following results.

THEOREM 2.3. *(See [6,14].) The function $\gamma(X)$ is a measure of weak noncompactness in the space L^1 such that $\beta(X) \leq \gamma(X) \leq 2\beta(X)$, where β denotes the DeBlasi measure of noncompactness. Moreover, $\gamma(B_{L^1}) = 2$.*

THEOREM 2.4. *(See [6,15].) Let X be nonempty, bounded, and compact in measure subset of L^1. Then $\chi(X) \leq \gamma(X) \leq 2\chi(X)$.*

In what follows, the following result will be of great importance for our purposes (cf. [14]).

THEOREM 2.5. *Let X be a bounded and compact in measure subset of L^1. If $F : X \to L^1$ is a continuous operator then it is weakly sequentially continuous on X.*

Observe now that joining the results contained in Theorems 2.1, 2.2, 2.4, and 2.5 we can deduce easily the following useful result.

COROLLARY 2.1. *Let Ω be a nonempty, bounded, closed, convex, and compact in measure subset of L^1. Assume that $F : \Omega \to \Omega$ is a continuous operator which is a contraction with respect to the measure of weak noncompactness γ. Then F has at least one fixed point in the set Ω.*

3. MAIN RESULT

According to the announcement given in the Introduction we will study the nonlinear Urysohn integral equation (1.1).

We assume that the functions involved in equation (1.1) satisfy the following hypotheses:

(i) $f \in L^1(\mathbb{R}_+)$ and is a.e. nonincreasing and positive on $\mathbb{R}_+$,

(ii) $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ satisfies the Carathéodory conditions i.e., the function $(t, s) \to u(t, s, x)$ is measurable for any fixed x and the function $x \to u(t, s, x)$ is continuous for almost all $(t, s) \in \mathbb{R}_+ \times \mathbb{R}_+$,

(iii) the function $t \to u(t, s, x)$ is a.e. nonincreasing on $\mathbb{R}_+$ for almost all $s \in \mathbb{R}_+$ and for each $x \in \mathbb{R}$,

(iv) the following inequality:

$$|u(t, s, x)| \leq q(t, s)[a(s) + b|x|]$$

is satisfied, for all $t, s \geq 0$ and $x \in \mathbb{R}$, where $a \in L^1(\mathbb{R}_+)$, $b \geq 0$ is a constant and $q : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a measurable function such that the operator

$$(Qx)(t) = \int_0^\infty q(t, s)x(s)\, ds$$

transforms the space L^1 into itself and is continuous.

Now, we can formulate our main result.

THEOREM 3.1. *Let Assumptions (i)–(iv) be satisfied. Assume additionally that $b|||Q||| < 1$, where $|||Q|||$ denotes the norm of the linear operator Q. Then, equation (1.1) has at least one solution x such that $x \in L^1$ and is a.e. nonincreasing on $\mathbb{R}_+$.*

PROOF. For $x \in L^1$, let us define two operators U and F by putting

$$(Ux)(t) = \int_0^\infty u(t, s, x(s))\, ds,$$
$$(Fx)(t) = f(t) + (Ux)(t).$$

Observe that in view of Assumption (ii) and the results concerning the so-called superposition operator [16], it follows that $s \to u(t, s, x(s))$ is a measurable function on $\mathbb{R}_+$, for $x \in L^1$ and for any fixed $t \geq 0$.

Now, we show that this function belongs to the space L^1 i.e., that the operator U transforms the space L^1 into itself.

Indeed, in view of our assumptions we get

$$\begin{aligned}
\int_0^\infty |(Ux)(t)|\, dt &= \int_0^\infty \left| \int_0^\infty u(t, s, x(s)\, ds \right| dt \\
&\leq \int_0^\infty \left(\int_0^\infty |u(t, s, x(s)|\, ds \right) dt \\
&\leq \int_0^\infty \left(\int_0^\infty (q(t, s)[a(s) + b|x(s)|])\, ds \right) dt \\
&= \int_0^\infty \left(\int_0^\infty q(t, s)a(s)\, ds \right) dt + b \int_0^\infty \left(\int_0^\infty q(t, s)|x(s)|\, ds \right) dt.
\end{aligned} \tag{3.1}$$

In view of our assumptions, we have that the functions $t \to \int_0^\infty q(t, s)a(s)\, ds$ and

$$t \to \int_0^\infty q(t, s)|x(s)|\, ds$$

belong to the space $L^1 = L^1(\mathbb{R}_+)$. This fact in conjunction with the above estimate implies that $Ux \in L^1$ and also $Fx \in L^1$.

Now, we show that the operator U is continuous on L^1.

To do this, let us take a sequence $(x_n) \subset L^1$ and a function $x \in L^1$ such that (x_n) converges to x in L^1, i.e.,

$$\lim_{n\to\infty} \int_0^\infty |x_n(t) - x(t)|\, dt = 0.$$

We show that

$$\lim_{n\to\infty} \int_0^\infty |(Ux_n)(t) - (Ux)(t)|\, dt = 0.$$

Thus, let us fix arbitrarily $T > 0$ big enough. We have

$$\begin{aligned}
&\int_0^\infty |(Ux_n)(t) - (Ux)(t)|\, dt \\
&= \int_0^T |(Ux_n)(t) - (Ux)(t)|\, dt + \int_T^\infty |(Ux_n)(t) - (Ux)(t)|\, dt \qquad (3.2)\\
&= \varepsilon_n + \int_T^\infty |(Ux_n)(t)|\, dt + \int_T^\infty |(Ux)(t)|\, dt,
\end{aligned}$$

where the sequence (ε_n) is defined by the formula

$$\varepsilon_n = \int_T^\infty |(Ux_n)(t) - (Ux)(t)|\, dt,$$

for $n = 1, 2, \ldots$.

Observe that if we consider the operator U on the space $L^1(0,T)$, then in view of Assumption (iv) it is majorized by the linear operator Q which we consider also on the space $L^1(0,T)$. Obviously, Q transforms the space $L^1(0,T)$ into itself. Moreover, Q is continuous since taking a sequence $(y_n) \subset L^1(0,T)$ and a function $y \in L^1(0,T)$, such that $y_n \to y$ (in the norm of $L^1(0,T)$) and extending y_n and y to the whole set $\mathbb{R}_+$ by putting $y_n(t) = y(t) = 0$, for $t \geq T$ and $n = 1, 2, \ldots$, we get that $y_n \to y$ in the norm of $L^1(\mathbb{R}_+)$. This implies that $Qy_n \to Qy$ in the norm of $L^1(\mathbb{R}_+)$ or equivalently, in the norm of $L^1(0,T)$. This means that Q transforms continuously $L^1(0,T)$ into itself. In view of the majorant principle [2], this assertion yields that the operator U acts continuously from the space $L^1(0,T)$ into itself. Thus, we infer that $\varepsilon_n \to 0$ as $n \to \infty$.

Further, let us notice that from the fact that $Ux \in L^1(\mathbb{R}_+)$ follows that the integral

$$\int_T^\infty |(Ux)(t)|\, dt$$

is arbitrarily small for T sufficiently large.

Next, we obtain

$$\begin{aligned}
\int_T^\infty |(Ux_n)(t)|\, dt &= \int_T^\infty \left| \int_0^\infty u(t,s,x_n(s))\, ds \right| dt \\
&\leq \int_T^\infty \left| \int_0^\infty |u(t,s,x_n(s))|\, ds \right| dt \\
&\leq \int_T^\infty \left(\int_0^\infty [q(t,s)a(s) + bq(t,s)|x_n(s)|]\, ds \right) dt \\
&\leq \int_T^\infty \left(\int_0^\infty q(t,s)a(s)\, ds \right) dt + b \int_T^\infty \left(\int_0^\infty q(t,s)|x_n(s) - x(s)|\, ds \right) dt \\
&\quad + b \int_T^\infty \left(\int_0^\infty q(t,s)|x(s)|\, ds \right) dt \\
&\leq \int_T^\infty \left(\int_0^\infty q(t,s)a(s)\, ds \right) dt + b \int_T^\infty \left(\int_0^\infty q(t,s)|x(s)|\, ds \right) dt
\end{aligned}$$

$$
\begin{aligned}
&+ b\int_0^\infty \left(\int_0^\infty q(t,s)|x_n(s)-x(s)|\,ds\right) dt\\
&\le \int_T^\infty \left(\int_0^\infty q(t,s)a(s)\,ds\right) dt + b\int_T^\infty \left(\int_0^\infty q(t,s)|x(s)|\,ds\right) dt\\
&+ b\int_0^\infty |(Q(|x_n-x|))(t)|\,dt\\
&\le \int_T^\infty \left(\int_0^\infty q(t,s)a(s)\,ds\right) dt + b\int_T^\infty \left(\int_0^\infty q(t,s)|x(s)|\,ds\right) dt\\
&+ b\|Q(|x_n-x|)\|_{L^1(\mathbb{R}_+)}\\
&\le \int_T^\infty \left(\int_0^\infty q(t,s)a(s)\,ds\right) dt + b\int_T^\infty \left(\int_0^\infty q(t,s)|x(s)|\,ds\right) dt\\
&+ |||Q|||\cdot\|x_n-x\|_{L^1(\mathbb{R}_+)}\\
&= \int_T^\infty ((Qa)(t))\,dt + \int_T^\infty ((Q|x|)(t))\,dt + b|||Q|||\cdot\|x_n-x\|_{L^1}.
\end{aligned}
$$

Now, keeping in mind our assumptions, from the above estimate, we deduce that for T sufficiently large and for n sufficiently large, the integral

$$\int_T^\infty |(Ux_n)(t)|\,dt$$

takes values arbitrarily small.

Combining this fact with the above obtained statements, in virtue of (3.2), we conclude that the operator U is continuous on the space L^1. Obviously, this implies that the operator F also acts continuously from the space L^1 to itself.

In what follows, take an arbitrary element $x \in L^1$. Then, taking into account estimate (3.1), we obtain

$$\|Fx\| \le \|f\| + \|Ux\| \le \|f\| + \|Qa\| + b|||Q|||\cdot\|x\|.$$

Since we have assumed that $b|||Q||| < 1$, from the above inequality we infer that the operator F transforms the ball B_r into itself for $r = (\|f\| + \|Qa\|)/(1 - b|||Q|||)$.

Now, let Ω stand for the subset of B_r consisting of all functions which are a.e. positive and nonincreasing on $\mathbb{R}_+$. Observe that in view of Theorem 2.2, the set Ω is compact in measure. Obviously, Ω is also nonempty, bounded, and convex. Moreover, we can also show that Ω is closed (cf. [5]).

It can be easily seen that the operator F transforms the set Ω into itself. Indeed, this assertion is a consequence of Assumptions (i) and (iii).

Now, let us fix a nonempty subset X of Ω. Further, take an arbitrary number ε and a measurable subset D of $\mathbb{R}_+$ with $m(D) \le \varepsilon$. Then, for an arbitrary fixed $x \in X$, we get

$$
\begin{aligned}
\int_D |(Fx)(t)|\,dt &\le \int_D f(t)\,dt + \int_D \left|\int_0^\infty u(t,s,x(s))\,ds\right| dt\\
&\le \int_D f(t)\,dt + \int_D \left(\int_0^\infty q(t,s)a(s)\,ds + b\int_0^\infty q(t,s)|x(s)|\,ds\right) dt\\
&\le \int_D f(t)\,dt + |||Q|||_D\cdot\|a\|_{L^1(D)} + b|||Q|||_D\cdot\|x\|_{L^1(D)}\\
&\le \int_D f(t)\,dt + |||Q|||\cdot\|a\|_{L^1(D)} + b|||Q|||_D\cdot\|x\|_{L^1(D)},
\end{aligned}
$$

where the symbol $|||Q|||_D$ denotes the norm of the linear operator Q acting from the space $L^1(D)$ into itself.

Hence, we have

$$\int_D |(Fx)(t)|\,dt \le \int_D f(t)\,dt + |||Q||| \int_D a(t)\,dt + b|||Q||| \int_D |x(t)|\,dt.$$

Consequently, we obtain

$$c(FX) \le b|||Q|||c(X), \tag{3.3}$$

since for any one-point subset Y of L^1 we have that $c(Y) = 0$.

Further, fixing arbitrarily $T > 0$ we obtain

$$\begin{aligned}
\int_T^\infty |(Fx)(t)|\,dt &\le \int_T^\infty f(t)\,dt + \int_T^\infty \left| \int_0^\infty u(t,s,x(s))\,ds \right| dt \\
&\le \int_T^\infty f(t)\,dt + \int_T^\infty \left(\int_0^\infty q(t,s)a(s)\,ds + b \int_0^\infty q(t,s)|x(s)|\,ds \right) dt \\
&\le \int_T^\infty f(t)\,dt + \|Qa\|_{L^1(T,\infty)} + b\|Qx\|_{L^1(T,\infty)} \\
&\le \int_T^\infty + |||Q||| \int_T^\infty a(t)\,dt + b|||Q||| \int_T^\infty |x(t)|\,dt.
\end{aligned}$$

Hence, keeping in mind that $d(Y) = 0$ for any singleton Y, we derive the following inequality:

$$d(FX) \le b|||Q|||d(X). \tag{3.4}$$

Combining (3.3) and (3.4), we arrive at the following inequality:

$$\gamma(FX) \le b|||Q|||\gamma(X).$$

This means that the operator F is a contraction with respect to the measure of weak noncompactness γ.

Finally, taking into account all facts established before and applying Corollary 2.1, we complete the proof.

REMARK 3.1. If we assume that the functions $f(t)$ and $t \to u(t,s,x)$ are a.e. nondecreasing and negative then applying the same argumentation, we can show that there exists a solution of equation (1.1) being a.e. negative and nondecreasing.

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